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LETTER TO THE EDITOR

On the possibility of obtaining any first integral of a Hamiltonian system via the Poisson theorem†

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Abstract. From two conveniently chosen first integrals of a Hamiltonian system it is possible (by taking Poisson brackets), to obtain all the local first integrals of the system.

It is well known [1, 2] that for Hamiltonian systems $\boldsymbol{X}_{\mathrm{H}}$ the Poisson bracket of two first integrals f_1 and f_2 of $\boldsymbol{X}_{\mathrm{H}}$ is again a first integral of $\boldsymbol{X}_{\mathrm{H}}$. This result is called the Poisson theorem. Of course the new first integral obtained in this way can be a function of f_1 and f_2 [3].

We prove here that choosing conveniently f_1 and f_2 we can generate a basis B of first integrals of $\boldsymbol{X}_{\mathrm{H}}$, such that any other integral of $\boldsymbol{X}_{\mathrm{H}}$ can be obtained from the first integrals of B.

In fact, let

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad i = 1, \ldots, n \tag{1}$$

be the Hamiltonian system. It is well known [2] that a local canonical transformation exists:

$$\boldsymbol{Q} = \boldsymbol{Q}(t; \boldsymbol{q}; \boldsymbol{p}) \qquad \boldsymbol{P} = \boldsymbol{p}(t, \boldsymbol{q}, \boldsymbol{p}) \qquad T = t \tag{2}$$

converting (1) into a Hamiltonian system such that in the new $(\boldsymbol{Q}, \boldsymbol{P})$ variables Hamilton's equations take the form:

$$\dot{\boldsymbol{Q}} = \mathbf{0} \qquad \dot{\boldsymbol{P}} = \mathbf{0}. \tag{3}$$

Accordingly $Q_1, \ldots, Q_n$; $P_1, \ldots, P_n$ form a basis of first integrals for equations (3).

Let us now assume that (3) has the property P of possessing two first integrals $\phi_0(\boldsymbol{Q}, \boldsymbol{P})$ and $\psi_0(\boldsymbol{Q}, \boldsymbol{P})$ generating (via Poisson brackets) any other first integral. If this is the case, then the functions ϕ and ψ defined (using (2)) by

$$\begin{aligned} \phi(t, \boldsymbol{q}, \boldsymbol{p}) &= \phi_0(\boldsymbol{Q}(t, \boldsymbol{q}, \boldsymbol{p}), \boldsymbol{P}(t, \boldsymbol{q}, \boldsymbol{p})) \\ \psi(t, \boldsymbol{q}, \boldsymbol{p}) &= \psi_0(\boldsymbol{Q}(t, \boldsymbol{q}, \boldsymbol{p}), \boldsymbol{P}(t, \boldsymbol{q}, \boldsymbol{p})) \end{aligned} \tag{4}$$

† Dedicated to Walter Thirring on his 60th birthday.

generate (via Poisson brackets) a basis of first integrals of equations (1). In fact, since (2) is a canonical transformation we can write:

$$\{\phi, \psi\}_{(t,q,p)} = \{\phi_0, \psi_0\}_{(t,Q(t,q,p),P(t,q,p))}. \tag{5}$$

Now, if $f(t, \boldsymbol{q}, \boldsymbol{p})$ is any first integral of equations (2) we have:

$$f(t, \boldsymbol{q}, \boldsymbol{p}) = F(\boldsymbol{Q}(t, \boldsymbol{q}, \boldsymbol{p}), \boldsymbol{P}(t, \boldsymbol{q}, \boldsymbol{p})) \tag{6}$$

F being a first integral of equations (3). But for the property P of ϕ_0 and ψ_0 we can write:

$$F = \mathscr{F}(\phi_0, \psi_0, \{\phi_0, \psi_0\}, \ldots) \tag{7}$$

and therefore, by (4) and (5):

$$f = \mathscr{F}(\phi, \psi, \{\phi, \psi\}, \ldots). \tag{8}$$

Therefore we have only to prove that equations (3) do indeed satisfy the property P. In other words it is enough to find two functions $\phi_0(\boldsymbol{Q}, \boldsymbol{P})$ and $\psi_0(\boldsymbol{Q}, \boldsymbol{P})$ such that:

$$\begin{aligned} Q_i &= \Phi_i(\phi_0, \psi_0, \{\phi_0, \psi_0\}, \ldots) \\ P_i &= \Pi_i(\phi_0, \psi_0, \{\phi_0, \psi_0\}, \ldots) \end{aligned} \qquad i = 1, 2, \ldots, n. \tag{9}$$

Let us see that the functions ϕ_0 and ψ_0 defined by:

$$\begin{aligned} \phi_0 &= P_1 \\ \psi_0 &= Q_2Q_1 + Q_3Q_1^2 + \ldots + Q_nQ_1^{n-1} + P_2Q_1^n + \ldots + P_nQ_1^{2n-2} \end{aligned} \tag{10}$$

enable us to prove the property P (for equations (3)). We assume $n > 1$; if $n = 1$ the problem is trivial since we can choose $\phi_0 = P$ and $\psi_0 = Q$.

Let us therefore prove that the functions ϕ_0 and ψ_0 defined in (10) generate, via Poisson brackets, the basis of first integrals $Q_1, \ldots, Q_n$; $P_1, \ldots, P_n$ of (3).

This can be immediately proved since we can write:

$$\frac{\partial f}{\partial Q_1} = \{f, P_1\} \tag{11}$$

for any function f depending solely on $P_1, \ldots, P_n$; $Q_1, \ldots, Q_n$.

On the other hand we can also write

$$\begin{aligned} \frac{\partial^{2n-2}\psi_0}{\partial Q_1^{2n-2}} &= (2n-2)!\,P_n \\ \frac{\partial^{2n-3}\psi_0}{\partial Q_1^{2n-3}} &= (\ldots)Q_1P_n + (2n-3)!\,P_{n-1} \\ \frac{\partial^{n}\psi_0}{\partial Q_1^{n}} &= (\ldots)Q_1^{n-2}P_n + \ldots + (\ldots)Q_1P_3 + P_2 \end{aligned} \tag{12}$$

where the symbols $(\ldots)$ stand for numerical coefficients irrelevant for our purposes.

By (11), with $f = \psi_0$ it follows that

$$\frac{\partial^j\psi_0}{\partial Q_1^j} = \left\{\frac{\partial^{j-1}\psi_0}{\partial Q_1^{j-1}}, P_1\right\} \qquad j = 1, 2, 3, \ldots. \tag{13}$$

Therefore from (13) and the first of equations (12) it is clear that P_n is generated, via Poisson brackets, from ϕ_0 and ψ_0.

On the other hand

$$\{\psi_0, P_n\} = \frac{\partial \psi_0}{\partial Q_n} \overset{(10)}{=} Q_1^{n-1} \tag{14}$$

and therefore Q_1^{n-1} (and as a consequence Q_1) is also generated via Poisson brackets (and functions of them). Accordingly, and from (12) and (13), we have obtained Q_1, $P_1, \ldots, P_n$ from ϕ_0 and ψ_0 via Poisson brackets (and functions of them).

In order to also get $Q_2, \ldots, Q_n$ in this way we introduce the function $\tilde{\psi}_0$ defined by:

$$\tilde{\psi}_0 = \psi_0 - P_2 Q_1^n - \ldots - P_n Q_1^{2n-2} = Q_2 Q_1 + Q_3 Q_1^2 + \ldots + Q_n Q_1^{n-1} \tag{15}$$

which is, by construction, obviously obtained from ϕ_0 and ψ_0 via Poisson brackets and functions of them.

We can write for $\tilde{\psi}_0$ an equation similar to (13):

$$\frac{\partial^j \tilde{\psi}_0}{\partial Q_1^j} = \left\{ \frac{\partial^{j-1} \psi_0}{\partial Q_1^{j-1}}, P_1 \right\} \qquad j = 1, 2, 3, \ldots. \tag{16}$$

When (16) is particularised for $j = 1$ we get:

$$\frac{\partial \tilde{\psi}_0}{\partial Q_1} = \{\psi_0, P_1\} = Q_2 \tag{17}$$

and therefore Q_2 can also be obtained from ϕ_0 and ψ_0 via Poisson brackets. Following the same procedure of putting $j = 2, 3, \ldots$ into (16) we finally obtain that $Q_2, Q_3, \ldots, Q_n$ are obtained from ϕ_0 and ψ_0 via Poisson brackets, as we desired to prove.

Summarising, any first integral of a Hamiltonian system can be obtained, via Poisson's theorem, by a successive and finite number of computations of the Poisson brackets of two conveniently chosen first integrals of $\boldsymbol{X}_{\mathrm{H}}$.

The authors want to point out that one of the referees has indicated that the above result is only true for first integrals depending explicitly on time.

On the other hand it is not an easy matter to give an example of the usefulness of the above result since in order to do this the local canonical transformation (2) should be computed.

References

[1] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison Wesley)

[2] Whittaker E 1959 *A Treatise on the Analytical Dynamics of Point and Rigid Bodies* 4th edn (Cambridge: Cambridge University Press)

[3] Pars L 1965 *A Treatise in Analytical Dynamics* (London: Heinemann)
Prange G 1901 *Encyklopädie der Mathematischen Wissenschaften* vol 4 (Leipzig: Teubner)